COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA 14 (3) (1994) 307–320

ON THE ASCENDING STAR SUBGRAPH DECOMPOSITION OF STAR FORESTS

KEJIE MA, HUISHAN ZHOU, and JIANQIN ZHOU

Received March 25, 1991
Revised December 1, 1993

Let G be a graph of size $\binom{n+1}{2}$ for some integer $n \geq 2$. Then G is said to have an ascending star subgraph decomposition if G can be decomposed into n subgraphs $G_1, G_2, \ldots, G_n$ such that each G_i is a star of size i with $1 \leq i \leq n$. We shall prove in this paper that a star forest with size $\binom{n+1}{2}$, possesses an ascending star subgraph decomposition if the size of each component is at least n, which is stronger than the conjecture proposed by Y. Alavi, A. J. Boals, G. Chartrand, P. Erdős and O. R. Oellermann.

1. Introduction

For definitions and notations not presented here, we follow [2]. We only consider simple undirected graphs, i.e., undirected graphs with no multiple edges and loops. The size of a graph is the number of edges of this graph.

Let G be a graph of size q, and let n be the positive integer with $\binom{n+1}{2} \leq q < \binom{n+2}{2}$. Then G is said to have an ascending subgraph decomposition (ASD), if G can be decomposed into n subgraphs $G_1, G_2, \ldots, G_n$ without isolated vertices such that G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \leq i \leq n-1$. Furthermore if each G_i is a star (matching, path, ..., etc.,), then G is said to have an ascending star (matching, path, ..., etc., respectively) subgraph decomposition or simply a star (matching, path, ..., etc. respectively) ASD.

Y. Alavi, A. J. Boals, G. Chartrand, P. Erdős and O. R. Oellermann proposed two conjectures [1]: Every graph of positive size has an ASD; a star forest of size $\binom{n+1}{2}$ with each component having size between n and $2n-2$ inclusively has a star ASD. In the same paper they have reduced the verification of the first conjecture to the following equivalent version: Every graph of size $\binom{n+1}{2}$ for $n = 1, 2, \ldots$ has an ASD. R. J. Faudree, A. Gyárfás and R. H. Schelp showed [5] that a star forest of size $\binom{n+1}{2}$ has an ASD, and that the complete graph K_{n+1} with $n+1$ vertices could be easily proved to have a star ASD and a path ASD. They further proved that any graph obtained from K_{n+1} by deleting any n edges has a star ASD. Some partial results are obtained for the second conjecture about the star ASD of a star forest of size $\binom{n+1}{2}$ in [3,4,7,9].

AMS subject classification code (1991): 05 C 70, 11 A 99, 90 C 35, 68 R 10

There are also some results about the matching ASD of a graph. Some of them put restrictions on the maximum degree of a graph. Two results were proved in [1]: a graph of size $\binom{n+1}{2}, n \geq 4$, with maximum degree at most 2, has a matching ASD; a forest of size $\binom{n+1}{2}$, with maximum degree d $(2 \leq 2d-2 \leq n)$, has a matching ASD. One was proved in [5]: a graph of size $\binom{n+1}{2}$, with maximum degree d $(n \geq 4d^2+6d+3)$, has a matching ASD. The other two can be found in [7]: Let G be a graph of size $\binom{n+1}{2}$. If G can be partitioned into n edge disjoint subgraphs G_i, $i = 1,2,\dots,n$, such that the size of G_i is i and for each $k \in \{2,3,\dots,n\}$, there is at most one edge of G_k which is incident with some vertex of the edge induced subgraph induced by the union of $G_1, G_2, \dots, G_{k-1}$, then G has a matching ASD. If G is a disconnected graph of n components, which have sizes $1,2,\dots,n-1$ and n, then G has a matching ASD. H. Fu obtained a result on the ASD of a graph with restrictions on the maximum degree too: a graph of size $\binom{n+1}{2}$ with maximum degree at most $(n-1)/2$ has an ASD [5], and a result on the ASD of complete bipartite graphs of size $\binom{n+1}{2}$ [6] which was also discovered independently and proved in different method by H. Zhou although not published.

In the discussion of ASD of a graph, we require that a smaller factor subgraph must be isomorphic to a proper subgraph of a greater factor subgraph. A closely related packing problem was obtained by loosening this requirement as considered in [8], where the author conjectured that the complete graph K_{n+1} can be decomposed into n edge disjoint trees of sizes $1,2,\dots,n$.

We now concentrate on one of the conjectures proposed in [1] and prove a stronger result: a star forest with size $\binom{n+1}{2}$, possesses a star ASD if the size of each component is at least n.

2. Main results

In order to prove the problem mentioned at the end of the last Section, we present and prove an equivalent number-theoretic problem following the presentations mentioned in [1]. We first introduce some new terminology and notation. Let $a, b_1, b_2, \dots, b_t$ be natural numbers where $b_1, b_2, \dots, b_t$ are all different. If $a = \sum_{i=1}^{t} b_i$, then a is said to be *decomposed to* $b_1, b_2, \dots, b_t$, and the set $\{b_1, b_2, \dots, b_t\}$ is called a decomposition of a, denoted by $a = [b_1, b_2, \dots, b_t]$. For natural numbers $a_1, a_2, \dots, a_k$ (not necessarily distinct), if each $a_i = [b_1^i, b_2^i \dots, b_{s_i}^i] (i = 1,2,\dots,k)$ is a decomposition, $\{b_1^i, b_2^i, \dots, b_{s_i}^i\} \bigcap \{b_1^j, b_2^j, \dots, b_{s_j}^j\} = \emptyset\}$ for $i \neq j$, and $\bigcup_{i=1}^{k} \{b_1^j, b_2^j, \dots, b_{s_j}^j\} = \{1,2,\dots,n\}$, then we simply say that $a_1, a_2, \dots, a_k$ is decomposed by $1,2,\dots,n$. Please note that the set here refers to multiset, i.e., the elements of sets are not necessarily different and unions are understood with multiplicity, i.e., $\{a\} \cup \{a\} = \{a, a\}$.

Our main result is the following theorem:

Theorem 2.1. *Assume that $a_i \geq n$ $(i = 1,2,\ldots,k)$ and $\sum_{i=1}^{k} a_i = \binom{n+1}{2}$. Then $a_1,a_2,\ldots,a_k$ can be decomposed by $1,2,\ldots,n$.*

In order to prove Theorem 2.1, we may further assume that $a_i > n$ $(i = 1,2,\ldots,k)$. Otherwise, say $a_k = n$, we only need to prove that $a_1,a_2,\ldots,a_{k-1}$ can be decomposed by $1,2,\ldots,n-1$. If $a_i > n$ $(i=1,2,\ldots,k)$, then $\frac{n(n+1)}{2} = \sum_{i=1}^{k} a_i > kn$, hence $n \geq 2k$. Therefore, we only need to prove the following equivalent theorem:

Theorem 2.2. *Assume that $a_i > n$ $(i=1,2,\ldots,k)$ and $\sum_{i=1}^{k} a_i = \binom{n+1}{2}$ where $n=2k+r$ $(r \geq 0)$. Then $a_1,a_2,\ldots,a_k$ can be decomposed by $1,2,\ldots,n$.*

We prove Theorem 2.2 by providing an algorithm in Section 3 and prove its correctness in Section 5.

3. The algorithm

We visualize our algorithm by working on a $2 \times (M-K)$ array $V = (v_{pi} : p = 1,2; i = 1,2,\ldots,M-K)$, called the decomposition model, where $M = \max\{a_1,a_2,\ldots,a_k\}$, $r = n-2k$, $K = k+r$, $N = 2K$, $v_{1i} = N+1-(M-i+1)$ and $v_{2i} = M-i+1$ for $1 \leq i \leq M-K$. See Figure 1 for a general illustration, and see Figure 2 for an illustration of a practical example.

V=	N+1+M	N+2+M	...	0	1	2	...	K-1	K
	M	M-1	...	N+1	N	N-1	...	K+2	K+1

Fig. 1

In Figure 1, each entry of V is also called a vertex. We will abuse the terminology by calling an entry of V, sometimes an integer, sometimes a vertex according to the context.

In the decomposition model $V = (v_{pi} : p=1,2; i=1,2,\ldots,M-K)$, $v_{1i}+v_{2i} = N+1$ for $i=1,2,\ldots,M-K$; $v_{1i}+v_{2j} < N+1$ for $i,j=1,2,\ldots,M-K$, and $i < j$; $v_{1i}+v_{2j} > N+1$ for $i,j=1,2,\ldots,M-K$, and $i > j$. For an integer $a = N+1$, we can draw an undirected edge $e = e(0) = e(a-N-1)$, called a *perpendicular edge,* between the vertex v_{1i} and v_{2i}. For an integer $a < N+1$, we can draw a directed edge $e = e(a-N-1)$, called an *up oblique edge,* from the vertex v_{2j} to the vertex v_{1i} with $j-i = N+1-a$. We call $v_{1(i+1)}, v_{1(i+2)}, \cdots, v_{1(j-1)}$ and v_{1j} the *up shadow vertices* of the edge e. For an integer $a > N+1$, we can draw a directed edge $e = e(a-N-1)$, called a *down oblique edge*, from the vertex v_{1i} to the vertex v_{2j} with $i-j = a-N-1$. We call $v_{2(j+1)}, v_{2(j+2)}, \cdots, v_{2(i-1)}$ and v_{2i} the *down shadow vertices* of the edge e. In all the above three cases, the edge $e(a-N-1) = v_{1i}v_{2j}$ represents a decomposition of a since $a=v_{1i}+v_{2j}, v_{1i} \neq v_{2j}$. The number $a-N-1$ is called a *positive, zero or negative term* depending on $a-N-1$ being greater than, equal to or less than zero. Once an edge e is drawn between v_{1i} and v_{2j}, both v_{1i} and v_{2j} are said to be saturated; otherwise they are unsaturated.

The idea of the algorithm is to decompose one integer into two distinct integers by drawing an edge (perpendicular, down oblique or up oblique, respectively) in the decomposition model V. If one of the two integers (i.e., the two end vertices of the edge) is still too big (i.e., greater than n), then we decompose this integer into two distinct integers by drawing an edge in the decomposition model again. If we can design the algorithm such that no integer from 1 to n will be used twice (i.e., no two edges will share a common vertex) and that the decomposition procedure cannot be infinite, then we will eventually decompose $a_1, a_2, \ldots, a_k$ by $1, 2, \ldots, n$.

Our algorithm has two procedures, trial decomposition and final decomposition, and a main program. In line 3 of the trial decomposition, a term $a-N-1$ was calculated for every integer a. S_{t0} is the set of zero terms, S_{t+} is the set of positive terms, and S_{t-} is the set of negative terms, where t is the control variable of repetitions. From line 4 to line 33, the integers are decomposed according to the following order: All integers with zero terms are decomposed right at the beginning in line 4 by drawing perpendicular edges from the right end to the left in the decomposition model V. Lines 5–19 are passed by since there are no shadow vertices now. An integer with the smallest positive term is decomposed first in 20–24, which generates down shadow vertices. Therefore, we go to lines 11–15 to decompose the integer with the largest negative term, which generates up shadow vertices. Then we go to lines 5–10 until we use up either the unsaturated up shadow vertices, or the integers with positive terms. Then followed in line 11,.... Always keep in mind that what we should do (lines 6–10) under the condition of having an unsaturated up shadow vertex in the first row and $S_{t+} \neq \emptyset$ (line 5) have the first priority; what we should do (lines 12–19) under the condition of having an unsaturated down shadow vertex in the second row and $S_{t-} \neq \emptyset$ (line 11) have the second priority; then follows the condition $S_{t+} \neq \emptyset$ (line 20) alone; and lastly the condition $S_{t-} \neq \emptyset$ (line 28) alone; i.e., line 21–26 have the third priority, and lines 29–33 have the lowest priority. Whenever the conditions for higher priority action occur we should do what follows. The graph obtained after running the trial decomposition on the set of integers is called a *trial graph*. In Section 5 we shall prove that the set of ending vertices of the down oblique edges falling in the set $C=\{n+1, n+2, \ldots, M\}$ will eventually be unchanged. Procedure 1 will be repeatedly run starting from $t=0$ until $B_t=B_{t-1}$ where t is the control variable (for definition of B_t, see line 34 of the algorithm). Therefore, we can run the final decomposition procedure to obtain the decomposition for each integer in the set A.

Algorithm

Input:

Given an integer n and a set $A=\{a_1, a_2, \ldots, a_k\}$ of integers with $a_i > n$ $(i=1, 2, \ldots, k)$ and $\sum_{k}^{i=1} a_i = \binom{n+1}{2}$.

Initialization:

Let $r = n - 2k, K = k + r$ and $N = 2K$. Construct a decomposition model $V = (v_{ij})_{2\times(M-K)}$ where $v_{1i} = N + 1 - (M - i + 1)$ and $v_{2i} = M - i + 1$ for $1 \le i \le M - K$. Let $C = \{n+1, n+2, \ldots, M\}$, $B_{-1} = \emptyset$, and $A_0 = A \cup B_{-1}$.

Procedure 1: Trial Decomposition

---

1. **input:** the decomposition model V with all the vertices of V unsaturated and set A_t;
2. **output:** the trial graph G_t and a set B_t.
3. Calculate $S_t = \{s | s = a - N - 1, a \in A_t\}$,
   $S_{t+} = \{s \in S_t | s > 0\}, S_{t-} = \{s \in S_t | s < 0\}$, and $S_{t0} = \{s \in S_t | s = 0\}$;
4. Draw $|S_{t0}|$ perpendicular edges from right to left in V, the end vertices of these edges become saturated;
5. **while** there is an unsaturated up shadow vertex in the first row and $S_{t+} \neq \emptyset$ **do**
6.   **Begin**
7. $s = \min\{s | s \in S_{t+}\}$;
8.   Draw a down oblique edge from the first unsaturated up shadow vertex from right in the first row such that there are s down shadow vertices, the end vertices of the down oblique edge become saturated;
9. $S_{t+} = S_{t+} \setminus \{s\}$;
10.   **end**
11. **if** there is an unsaturated down shadow vertex in the second row and $S_{t-} \neq \emptyset$ **then**
12.   **begin**
13. $s = \max\{s | s \in S_{t-}\}$;
14.   Draw an up oblique edge from the first unsaturated down shadow vertex from right in the second row such that there are $|s|$ up shadow vertices, the end vertices of the up oblique edge become saturated;
15. $S_{t-} = S_{t-} \setminus \{s\}$;
16.   **if** there is an unsaturated up shadow vertex in the first row **then**
17.   **go to 5;**
18.   **else go to 11;**
19.   **end**
20. **else if** $S_{t+} \neq \emptyset$ **then**
21.   **begin**
22. $s = \min\{s | s \in S_{t+}\}$;
23.   Draw a down oblique edge from the first unsaturated vertex from right in the first row such that there are s down shadow vertices; the end vertices of the down oblique edge become saturated
24. $S_{t+} = S_{t+} \setminus \{s\}$;
25.   **go to 11;**
26.   **end**
27. **else**
28. **while** $S_{t-} \neq \emptyset$ **do**
29.   **begin**

$s = \max\{s | s \in S_{t-}\}$;
Draw an up oblique edge from the first unsaturated vertex from right in the second row such that there are $|s|$ up shadow vertices; the end vertex of the up oblique edge become saturated
$S_{t-} = S_{t-} \setminus \{s\}$;
end
$B_t =$ the intersection of C and the set of ending vertices of the down oblique edges in the trial graph G_t.

Procedure 2: Final Decomposition

---

for $a \in A$ **do**
begin
if the edge of the term $a - N - 1$ has two endvertices a^1 and a^2 with $a^1 < a^2$ **then**
$a = [a^1, a^2]$;
while $a^{2i} > n$ **do**
begin
if the edge of the term $a^{2i} - N - 1$ has two endvertices a^{2i+1} and a^{2i+2} with $a^{2i+1} < a^{2i+2}$ **then**
$a^{2i} = [a^{2i+1}, a^{2i+2}]$;
$i = i + 1$;
end
end

Main

$t = 0$;
call trial decomposition (A_0, B_0);
while $B_t \neq B_{t-1}$ **do**
begin
$A_{t+1} = A \cup B_t$;
call trial decomposition (A_{t+1}, B_{t+1});
$t = t + 1$;
end
call final decomposition

V=	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12
	31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	16	15	14	13

Fig. 2

4. Example

In order to better understand our algorithm and the proof in Section 5, we provide an example as follows: To find the decomposition of

$$A = \{31,30,29,29,26,22,22,21\}$$

by $1,2,\ldots,20$. We know that $n = 20$, $k = 8$, $r = 4$, $K = 12$, $N = 24$, $M = 31$, and the decomposition model is Figure 2. The set of terms is $S_0 = \{-4,-3,-3,1,4,4,5,6\}$. $S_{00} = \emptyset$, $S_{0+} = \{1,4,4,5,6\}$, $S_{0-} = \{-4,-3,-3\}$ and $C = \{21,22,23,24,25,26,27,28,29,30,31\}$.

(0) In running the trial decomposition for A on the decomposition model V, we pass by lines 4–19 at the beginning since $S_{00} = \emptyset$, and there is neither unsaturated up shadow vertex in the first row nor unsaturated down shadow vertex in the second row. Draw a down oblique edge from vertex 12 to 14 by lines 20–24; go to line 11 by line 25; draw an up oblique edge from vertex 13 to 9 by going over lines 12–15; go to line 5 after checking line 16; repeatedly draw down oblique edges from vertex 11 to 18, and from vertex 10 to 19 by going over line 5–10 twice; then follow lines 11–15 to draw an up oblique edge from vertex 15 to 7, there is one unsaturated up shadow vertex 8 now; go to line 5; draw a down oblique edge from 8 to 22 by lines 6–10, there are unsaturated down shadow vertices 16, 17,20, and 21 now; follow lines 11–15 to draw an up oblique edge from vertex 16 to 5, there is an unsaturated up shadow vertex 6; go to lines 5–10 to draw a down oblique edge from 6 to 25. Now S_0 is empty, and the trial decomposition is stopped. See Figure 3 from the trial graph G_0. The number by the side of the edge is the order by which that edge is drawn. $B_0 = \{25,22\}, 31 = 25+6, 30 = 22+8$. The integers 25 and 22 are still too big (> 20). Therefore, in the next run of trial decomposition, we need to decompose the set $A_1 = A \cup B_0$.

G_0=

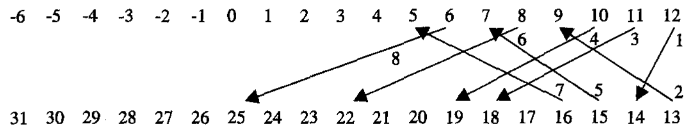

Fig. 3

(1) Do the trial decomposition of $A_1 = A \cup B_0$ on the decomposition model V to obtain the trial graph G_1 as in Figure 4. Now the set of terms is $S_1 = S_0 \cup \{-3,0\}$. $B_1 = \{27,23\}$.

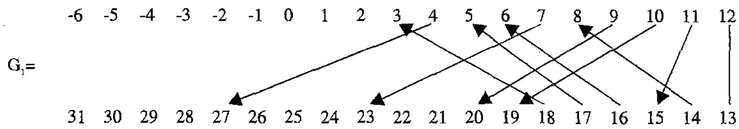

Fig. 4

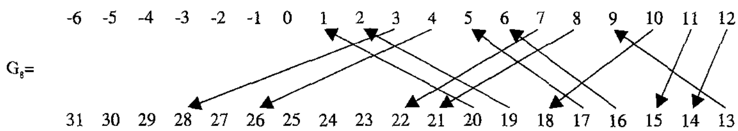

Fig. 5

Continuing to do the trial decomposition, we have $B_2 = \{27,24,21\}$, $B_3 = \{27,25,21\}$, $B_4 = \{27,25,22\}$, $B_5 = \{28,25,22\}$, $B_6 = \{28,26,22\}$, $B_7 = \{28,26,22,21\}$, $B_8 = \{28,26,22,21\}$. The last trial graph is Figure 5.

Therefore, $21 = [1,20]$, $22 = [5,17]$, $22 = [6,16]$, $26 = [12,14]$, $29 = [8,21]$, $29 = [7,22]$, $30 = [4,26]$, $31 = [3,28]$; and $21 = [2,19]$, $22 = [9,13]$, $26 = [11,15]$, $28 = [10,18]$. And we have the final solution after running the final decomposition as follows: $21 = [1,20]$, $22 = [5,17]$, $22 = [6,16]$, $26 = [12,14]$, $29 = [8,2,19]$, $29 = [7,9,13]$, $30 = [4,11,15]$, $31 = [3,10,18]$.

5. The proof

Recall that a number in S_{t+}(S_{t-} and S_{t0} respectively) is called a positive (negative and zero, respectively) term of the trial graph G_t. In the trial graph G_t, there is a down oblique (up oblique and perpendicular, respectively) edge $e_t(s)$ corresponding to each positive (negative and zero, respectively) term. Some properties are provided for the trial graph G_t in the following in order to prove the correctness of our algorithm. There are both directed and undirected edges in G_t. But the direction of edges are ignored for the underlying graph of G_t. The following Lemma 5.1 is an obvious observation.

Lemma 5.1. *The edge set of the underlying graph of the trial graph is an edge independent set.* ∎

The algorithm of the trial decomposition works on a set of numbers $A \cup B$, or equivalently on a set $S = \{s : s = a - N - 1 \text{ for } a \in A \cup B\}$, to obtain a trial graph G. Once the set $A \cup B$ or S is fixed, the trial graph G is completely determined. In the proof of following lemmas we should always bear the following facts in mind. We always draw the edges with zero terms first. Among edges with positive terms, those with smaller terms were drawn first. Among edges with negative terms, those with greater terms (i.e., it's absolute value is smaller) were drawn first.

Lemma 5.2. *Let the two trial graphs G_i and G_h have the same positive terms. Let G_h have one negative or zero term more than G_i. Then for any positive terms s, the down oblique edge $e_h(s)$ of G_h is either at the same location as or to the left of the corresponding down oblique edge $e_i(s)$ of G_i. (Note: We use the notation $e_i(s)$ to mean that this edge is in the trial graph G_i. When we compare the location of two edges in two different trial graphs, we compare them in the same decomposition model.)*

Proof. Assume that G_i have negative and zero terms as follows

$$s_m \leq s_{m-1} \leq \ldots \leq s_j \leq \ldots \leq s_1 \leq 0.$$

Assume that G_h has one term c more than $G_i (c \leq 0)$. Without loss of generality, let $s_{j+1} < c \leq s_j$. Then the negative and zero terms in G_h in the order from the smallest to the greatest are

$$u_{m+1} \leq u_m \leq \cdots \leq u_{j+2} \leq u_{j+1} \leq u_j \leq \ldots \leq u_1.$$

where $u_1 = s_1, \ldots, u_j = s_j, u_{j+1} = c, u_{j+2} = s_{j+1}, \ldots u_m = s_{m-1}$, $u_{m+1} = s_m$. Since G_i and G_h have the same positive terms, same negative and zero terms which are greater than or equal to $s_j = u_j$, then the starting vertex of the up oblique edge of s_{j+1} in G_i must be the same as the starting vertex of the up oblique edge of $u_{j+1} = c$ in G_h by the algorithm. Hence the ending vertex (denoted by $ev(u_{j+1})$) of the up oblique edge of $c = u_{j+1}$ is at the right side of the ending vertex $ev(s_{j+1})$ of the up oblique edge of s_{j+1} since $s_{j+1} < c \leq 0$. Therefore, the down oblique edge starting at $ev(u_{j+1})$ and at the left side of $ev(u_{j+1})$ in G_i will either move to the left by one vertex or unmoved in G_h ∎

Lemma 5.3. *Let G_f have one negative term c more than G_i, and G_h one negative or zero term $c+1$ more than G_i. Then for any positive term s, the corresponding down oblique edge $e_h(s)$ of G_h is either at the same location as or to the left of the corresponding down oblique edge $e_f(s)$ of G_f.*

Proof. Assume that G_i have negative and zero terms as follows

$$s_m \leq s_{m-1} \leq \ldots \leq s_{j+1} \leq s_j \leq \ldots \leq s_1 \leq 0.$$

Assume that

$$s_{j+1} \leq c < c+1 \leq s_j.$$

Since G_f and G_h have the same positive terms, and the same negative and zero terms which are greater than or equal to s_j, the starting vertex of the up oblique edge of c in G_f must be the same as the starting vertex of the up oblique edge of $c+1$ (or perpendicular edge if $c+1=0$) in G_h. Hence the ending vertex $ev(c+1)$ of the up oblique edge of $c+1$ in G_h is at the right side of the ending vertex $ev(c)$ of the up oblique edge of c in G_f. Therefore, the down oblique edge starting at $ev(c+1)$ in G_f will move left by one vertex in G_h. Other down oblique edge in G_f remain unchanged in G_h. ∎

Lemma 5.4. *Let G_h have one positive term c more than G_i. Then for any positive term s except c, the corresponding down oblique edge $e_h(s)$ of G_h is either at the same location as or to the left of the corresponding down oblique edge $e_i(s)$ of G_i.* ∎

We omit the proof since it is similar to and simpler than the above proofs.

Lemma 5.5. *Let G_f have one positive or zero term d more than G_i, and G_h one positive term $d+1$ more than G_i. Then the ending vertex of the p-th down oblique edge of G_h is either at the same location as or to the left of the ending vertex of the p-th down oblique edge of G_f.*

Note. The down oblique edges can be ordered from either side in this lemma. But in the following definition of set function f generated by the trial decomposition we use the order from the left: The first down oblique edge is the one with the largest term.

Proof. Assume that G_i have positive and zero terms as follows

$$s_m \geq s_{m-1} \geq \ldots \geq s_{j+1} > s_j \geq \ldots \geq s_1 \geq 0$$

and assume that

$$s_{j+1} \geq d+1 > d \geq s_j.$$

The presentation in the following disregard the case $d=0$. It is reasonable to omit it since there is only very minor differences in wording for the case $d=0$. Since G_f and G_h have the same negative and zero terms, and the same positive terms which are smaller than or equal to $s_j, e_f(s)$ and $e_h(s)$ must have the same location for $s = s_1, s_2, \ldots, s_{j-1}$ and s_j and the starting vertex of the down oblique edge of $d+1$ in G_h must be the same as the starting vertex of the down oblique edge of d in G_f. Hence the ending vertex $ev(d+1)$ of the down oblique edge of $d+1$ in G_h moves left by one vertex to the ending vertex $ev(d)$ of the down oblique edge of a in G_f. See Figure 6 for the illustration.

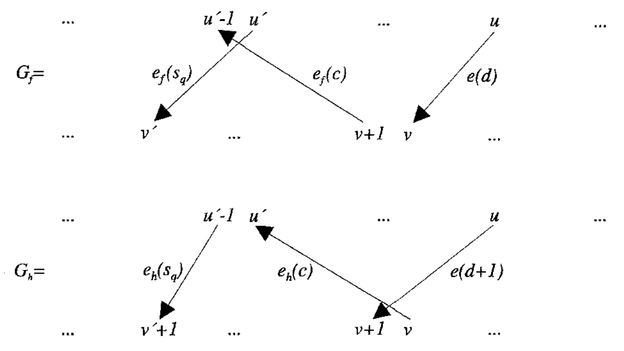

Fig. 6

Assume that the negative term c determine an up oblique edge $e_h(c)$ starting at $ev(d)$ in G_h; and determine an up oblique edge $e_f(c)$ starting at $ev(d+1)$ in G_f. Then $e_f(c)$ is one vertex left to $e_h(c)$. Assume that the positive term s_q determined a down oblique edge $e_f(s_q)$ starting at the ending vertex of $e_h(c)$ in G_f. Then the down oblique edge $e_h(s_q)$ determined by s_q in G_h must start at the ending vertex of $e_f(c)$ which is one vertex left to the ending vertex of $e_h(c)$.

The down oblique edge $e_h(s_p)$ for $s_q > s_p > d+1$ has the same position as the down oblique edge $e_f(s_p)$. The down oblique edge $e_h(s_p)$ for $s_p \geq s_q$ is either at the same location as or to the left of the down oblique edge $e_f(s_p)$. ∎

We will use the following definition to both simplify and formalize the statements of Lemma 5.2 through 5.5, and their generalizations. We define an ordering among sets of integers by $A \succ B$ or $B \prec A$ if $A = \{a_1, a_2, \ldots, a_l\}$, $B = \{b_1, b_2, \ldots, b_k\}$, $a_1 \geq a_2 \geq \ldots \geq a_l$, $b_1 \geq b_2 \geq \ldots \geq b_m$, $l \geq m$, $a_1 \geq b_1$, $a_2 \geq b_2$, $\ldots$, $a_m \geq b_m$. Here and in the following, when we write a set $A = \{a_1, a_2, \ldots, a_l\}$, we always assume that the list is in the nonincreasing order, i.e., $a_1 \geq a_2 \geq \ldots \geq a_l$. Let T_n denote the "cut at n" map, i.e., $T_n(A) = \{a \in A : a > n\}$. The following lemma is obvious.

Lemma 5.6. (i) *$A \cup C \succ A$;*
(ii) *If $A \succ B$ and C is any set of integers, then $A \cup C \succ B \cup C$;*
(iii) *If $A \succ B$, then $T_n(A) \succ T_n(B)$.* ∎

In the algorithm, for any set of integers $A = \{a_1, a_2, \ldots, a_u\}$ where $u \geq k$, and $n < a_i \leq M$, let $G(A)$ to be the graph obtained by applying the trial decomposition, $f(A)$ to be the set of end vertices of down oblique edges in $G(A)$. From Lemma 5.2 and 5.4 we can have "$f(A \cup \{s\}) \succ f(A)$", and Lemma 5.3 and 5.5 can be directly translated as "$f(A \cup \{s+1\}) \succ f(A \cup \{s\})$". For any two sets A and B with $A \succ B$, we can insert a finite chain of sets between B and A such that in each step we either add a new element or increase the value of one element by one. By repeatedly applying the above two results we have the following theorem.

Theorem 5.7. *If $A \succ B$, then $f(A) \succ f(B)$, and $T_n(f(A)) \succ T_n(f(B))$.* ∎

In order to prove Theorem 2.2 we only need to prove the following theorem.

Theorem 5.8. *Under the conditions of Theorem 2.2, the algorithm gives a decomposition of $a_1, a_2, \ldots, a_k$ by $1, 2, \ldots, n$.*

Proof. Recall that in each trial decomposition of the algorithm, among edges with positive terms, those with smaller values were drawn first, hence the numbers of down shadow vertices of consecutive down oblique edges added to the model are not decreasing, so are the numbers of up shadow vertices of consecutive up oblique edges. Therefore, different down oblique edges are neither intersecting each other nor sharing common end vertices, the same property holds for the up oblique edges. It is now obvious that each integer appears in the decomposition at most once.

Denote, by G_0, the trial graph obtained by applying the trial decomposition to $A_0 = A \cup B_{-1} = \{a_1, a_2, \ldots, a_k\}$. Denote, by B_0, the set of ending vertices of down oblique edge of G_0 in $C = \{n, n+1, \ldots, M\}$, i.e., $B_0 = T_n(f(A \cup B_{-1})) = T_n(f(A_0))$. If $B_0 = \emptyset$, then apply the final decomposition.

If $B_0 \neq \emptyset$, then apply the trial decomposition to $A_1 = A \cup B_0$ to obtain the trial graph G_1, and the set $B_1 = T_n(f(A \cup B_0)) = T_n(f(A_1))$. By Lemma 5.6 and Theorem 5.7, $B_1 \succ B_0$.

If $B_1 = B_0$, then apply the final decomposition; otherwise apply the trial decomposition to $A_2 = A \cup B_1$ to obtain the trial graph G_2, and the set $B_2 = T_n(f(A \cup B_1)) = T_n(f(A_2))$. By Lemma 5.6 and Theorem 5.7, $B_2 \succ B_1$.

Similarly we obtain a series of sets

$$B_0 \prec B_1 \prec B_2 \prec \ldots \prec B_i \prec B_{i+1} \prec \ldots$$

We shall now prove by contradiction that $|B_i| \leq r$ for $i = 0, 1, 2, \ldots$.

Assume that $B_{t-1} = \{b_1, b_2, \ldots, b_u\}$, $u \leq r$, $B_t = T_n(f(A_t)) = \{c_1, c_2, \ldots, c_u, c_{u+1}, \ldots, c_v\}$, and that $v > r$, where $A_t = A \cup B_{t-1}$.

Let $B'_t = \{c_1, c_2, \ldots, c_u, c_{u+1} \ldots, c_r\}$. We apply the trial decomposition to $A \cup B'_t$ to obtain the trial graph G, and the set

$$T_n(f(A \cup B'_t)) = \{d_1, d_2, \ldots, d_u, \ldots, d_r, d_{r+1}, \ldots, d_{v_1}\}$$

Then $v_1 \geq v$ and $d_i \geq c_i$ for $1 \leq i \leq r$. The reasons of $v_1 \geq v$ are from Lemma 5.6, Theorem 5.7 and the fact that $\{c_1, \ldots, c_u, \ldots, c_r\} \succ \{b_1, \ldots, b_u\}$.

During the construction of the trial graph G, a vertex of $\{K+1, \ldots, n, \ldots, M\}$ is saturated after decomposing a number in $A \cup \{c_1, c_2, \ldots, c_r\}$. Since the number of down oblique edges whose ending vertices belong to C in G is $v_1 \geq v > r$, the number of up and down oblique edges who start and end in $\{K+1, K+2, \ldots, n\}$ in G is $\leq k+r-v_1 < k$, i.e., there are at least $n-K-(k+r-v_1) = v_1 - r > 0$ (because $n = K+k$) unsaturated down oblique shadow vertices in $\{K+1, K+2, \ldots, n\}$. Therefore, there are no up oblique edge which starts in C by our algorithm. It turns out that the number of up and down oblique edges who start and end in $\{K+1, K+2, \ldots, n\}$ in G is exactly $k+r-v_1$, i.e., there are exactly $n-K-(k+r-v_1) = v_1 - r$ unsaturated down oblique shadow vertices in $\{K+1, K+2, \ldots, n\}$. Therefore, the starting vertex of the most left up oblique edge e, say $e = e(a_m - N - 1)$, is smaller than n. Since the corresponding number a_m is greater than n, then the ending vertex of e is greater than 1. Hence all the ending vertices of up oblique edges belong to $\{1, 2, \ldots, K\}$. All the starting vertices of down oblique edges belong to $\{1, 2, \ldots, K\}$ since the number of all edges is $k+r = K$. Therefore, every number in $\{1, 2, \ldots, K\}$ is saturated.

We now define a set-valued function h from the set $A \cup C$ to the set $\{1, 2, \ldots, M\}$. (Recall that members in $A \cup C$ are not necessarily distinct in value). If a number $a \in A \cup \{c_1, c_2, \ldots, c_u, \ldots, c_r\}$, then a is decomposed into two numbers a^1 and $a^2 \in \{1, 2, \ldots, M\}$, $a = a^1 + a^2$, and we set $h(a) = \{a^1, a^2\}$. If $a \in \{n+1, \ldots, M\} \setminus (\{c_1, \ldots, c_r\} \cup \{d_1, \ldots, d_{v_1}\})$, then set $h(a) = a$. Let $h(d_i) = c_i$ for $1 \leq i \leq r$ and $d_i \in \{d_1, \ldots, d_r\} \setminus (\{c_1, \ldots, c_r\} \cap \{d_1, \ldots, d_r\})$. Then $d_i \geq h(d_i)$ for $1 \leq i \leq r$ and $d_i \in \{d_1, \ldots, d_r\} \setminus (\{c_1, \ldots, c_r\} \cap \{d_1, \ldots, d_r\})$. For any number $a \in \{d_{r+1}, \ldots, d_{v_1}\}$, we set $h(a)$ to be an unsaturated vertex in $\{K+1, \ldots, n\}$. There is a one to one correspondence between $\{d_{r+1}, \ldots, d_{v_1}\}$ and all the unsaturated vertices in $\{K+1, \ldots, n\}$ since there are exactly $v_1 - r$ unsaturated vertices in $\{K+1, \ldots, n\}$. Therefore, $d_i > n \geq h(d_i)$ for $r+1 \leq i \leq v_1$.

Obviously if a runs over $A \cup C$ once, then $h(a)$ runs over $\{1, 2, \ldots, M\}$ once. $a = a^1 + a^2$ if $h(a) = \{a^1, a^2\}$. $a \geq h(a)$ for other cases among which there are $v_1 - r$ strict inequalities. Therefore,

$$\sum_{i=1}^{k} a_i + \sum_{j=n+1}^{M} j > 1 + 2 + \ldots + M$$

contradicting to the fact that $\sum_{i=1}^{k} a_i = 1+2+\ldots+n$. Therefore, $|B_i|$ cannot be greater than r for $i=0,1,2,\ldots$.

Since $|A \cup B_i| \leq k+r=K$, The starting vertex of the edge which represents the decomposition of a number in $A \cup B_i$ is not smaller than 1, hence the ending vertex of that edge is not greater than $a_k-1<M$. Since $B_i(i=0,1,2,\ldots)$ is nondecreasing, there exists a number t such that $B_t=B_{t+1}$. That is, the trial decomposition of the algorithm will eventually terminate in finite steps.

For any number $a \in A, a=a^1+a^2$ if the two end vertices of the edge $e=e(a-N-1)$ is a^1 and a^2 in the trial graph G_t. If both a^1 and a^2 are not greater than n, then we are done. If one of a^1 and a^2, is greater than n, then $a^2 \in B_{t+1}=B_t$. a^2 was decomposed again in the trial graph G_t, say $a^2=a^3+a^4$. If both a^3 and a^4 are not greater than n, then we are done: $a=a^1+a^3+a^4$. If one of a^3 and a^4, is greater than n, then $a^4 \in B_{t+1}=B_t$. a^4 was decomposed again in the trial graph G_t, say $a^4=a^5+a^6$. If both a^5 and a^6 are not greater than n, then we are done: $a=a^1+a^3+a^5+a^6$. Otherwise we continue our decomposition until each number is not greater than n. We can apply the above decomposition to each member of A, and write each integer a_i as sums of positive integers $\leq n$. We use any positive integer $\leq n$ at most once since no two edges share the common vertices in the trial graph. As $a_1+a_2+\ldots+a_k=1+2+\ldots+n$ we have to use all integers from 1 to n exactly once. Therefore, A is decomposed by $1,2,\ldots,n$. ∎

Note that every integer a_i is decomposed by at most $r+2=n-2k+2$ summands since $|B_t| \leq r$, and in the worst case a_i is decomposed by at most r times.

Acknowledgment

We owe our thanks to Zukang Xu for helping in computer implementation, and the referees for helpful comments.

References

[1] Y. Alavi, A. J. Boals, G. Chartrand, P. Erdős, and O. R. Oellermann: The ascending subgraph decomposition problem, *Congressus Numerantium* **58** (1987), 7–14.

[2] G. Chartrand, and L. Lesniak: *Graphs and Digraphs*, 2nd edition, Wadsworth and Brooks/Cole, Monterey, CA (1986).

[3] H. Chen, and K. Ma: On a new subgraph decomposition problem, *Utilitas Mathematica* **37** (1990), 265–270.

[4] H. Chen, K. Ma, and H. Zhou: The ascending star subgraph decomposition of star forests, accepted by Ars Combinatoria, to appear.

[5] R. J. Faudree, A. Gyárfás, and R. H. Schelp: Graphs which have an ascending subgraph decomposition, *Congressus Numerantium* **59** (1987), 49–54.

[6] H. Fu: A note on the ascending subgraph decomposition, *Discrete Mathematics*, to appear.

[7] H. Fu: Some results on ascending subgraph decomposition, *Bulletin of the Institute of Mathematics Academia Sinica*, **16** (1988) (4), 315–319.

[8] A. GYÁRFÁS, and J. LEHEL: Packing trees of different order into K_n, Colloquia Mathematica Societatis János Bolyai 18, *Combinatorics*, Keszthely, Hungary, 1976, 463–469.

[9] F. ZHAO: On a new subgraph decomposition problem, *Qufu Shifan Daxue Xuebao*, **14** (1988) (4), 58–61.

Kejie Ma

Institute of Operations Research,
Qufu Normal University,
Qufu, Shandong,
P. R. China.

Jianqin Zhou

Department of Mathematics,
Qufu Normal University,
Qufu, Shandong,
P. R. China

Huishan Zhou

Department of Mathematics and Computer Science
Georgia State University,
Atlanta, Georgia, 30303–3083
USA
mathhz@gsusgi2.gsu.edu
and
Institute of Operations Research
Qufu Normal University
Qufu, Shandong,
P. R. China